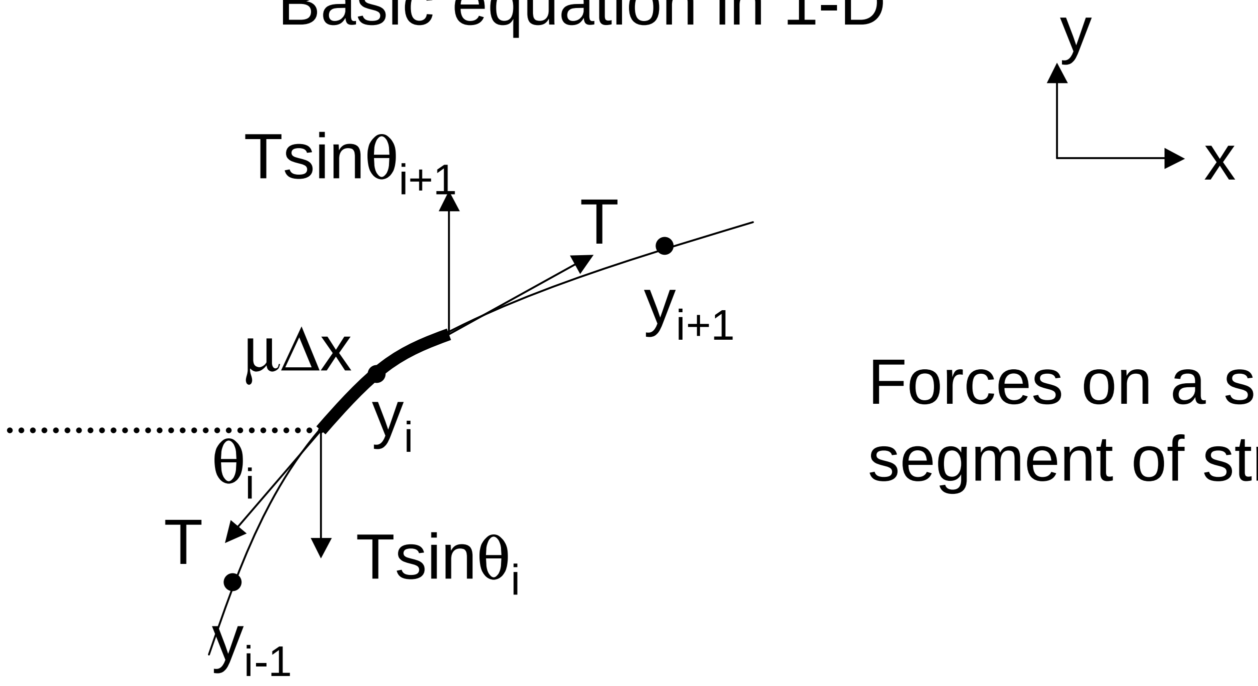
Waves and Music

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Basic equation in 1-D



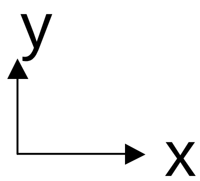
Forces on a short segment of string

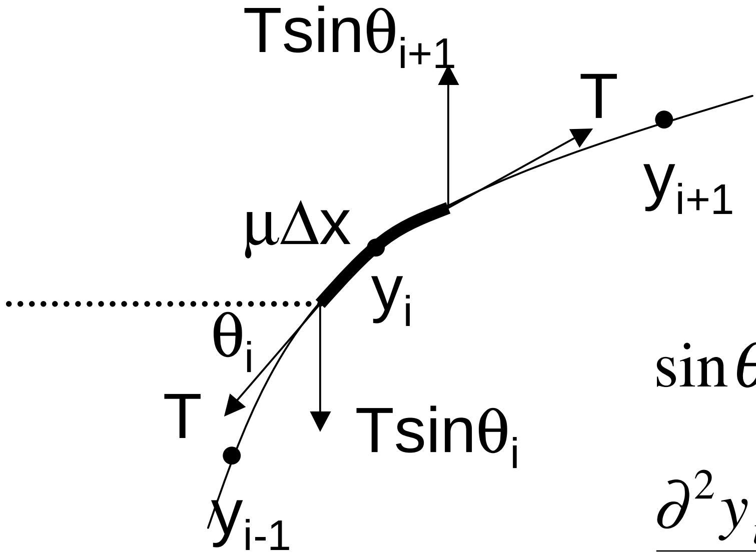
The force on segment i comes from the neighboring segments.

$$(\mu\Delta x)\frac{d^2 y_i}{dt^2} = T\sin\theta_{i+1} - T\sin\theta_{i-1}$$

Waves on a string

Basic equation in 1-D





$$\sin\theta_{i+1} \approx \frac{y_{i+1} - y_i}{\Delta x} \qquad \sin\theta_i \approx \frac{y_i - y_{i-1}}{\Delta x}$$

$$\frac{\partial^2 y_i}{\partial t^2} \approx \left(\frac{T}{\mu}\right)\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} \approx \left(\frac{T}{\mu}\right)\frac{\partial^2 y_i}{\partial x^2}$$

Solutions are functions of the form:

$$y = f(x \pm ct) \qquad c = \sqrt{\frac{T}{\mu}}$$

Waves on a string

Treat x,t as discrete variables: $x = i\Delta x \quad t = n\Delta t$

Then using our well known approximation:

$$\frac{y(i,n+1) - 2y(i,n) + y(i,n-1)}{(\Delta t)^2} \approx c^2 \left[\frac{y(i+1,n) - 2y(i,n) + y(i-1,n)}{(\Delta x)^2} \right]$$

Assuming we know the position of the string at time n-1,n, then we can calculate at time n+1. Rearranging:

$$y(i,n+1) = 2[1 - r^2]y(i,n) - y(i,n-1) + r^2[y(i+1,n) + y(i-1,n)]$$

where

$$r \equiv \frac{c\Delta t}{\Delta x}$$

Waves on a string

How to treat the ends of the string ? Different possibilities - start with ends of string fixed:

$$y(0,n) = y(M,n) = 0 \quad \forall n \qquad \mathrm{M}+1 \text{ grid points}$$

We are not allowed to modify these endpoints.

Example: start with a Gaussian pulse at the center of the string. Take

$$\Delta x = 0.01 \quad M = 400 \quad \Delta t = 0.01 \quad c = 1$$

This yields $r = 1$, and the string length is 4 (somethings)

→ See 'movie'

Waves on a string

The simulation looks quite accurate. How accurate is it ? Difficulty here is that there are two types of steps, Δt, Δx. We spefically chose the values so that r=1. Why ?

The information from our approximation only moves one grid point at a time (information from i shared with i-1,i+1). If the speed of the wave is great than this, then the algorithm cannot keep up and the result is a diverging series.

Let's look in more detail - von Neumann analysis and Courant condition

Partial Differential Equations

General Classes:

- hyperbolic, e.g., wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

- Parabolic, e.g., diffusion equation

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(D \frac{\partial y}{\partial x} \right)$$

- elliptic, e.g., Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y)$$

Boundary value problems: wave & diffusion equation need information on y at some t, then can propagate. Poisson equation needs information on a spatial boundary, and static slotuion for interior points determined.

Partial Differential Equations

We have seen examples of initial and boundary value problems. The boundary value problems are typically stable, and the task is usually to optimize speed and memory allocation.
For initial value problems, stability is a key issue as we have seen before.
The 1-D wave equation is a *flux-conservative equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{can be rewritten as a set of two equations:}$$

$$\frac{\partial r}{\partial t} = c\frac{\partial s}{\partial x} \qquad \frac{\partial s}{\partial t} = c\frac{\partial r}{\partial x} \qquad \text{with } r \equiv c\frac{\partial u}{\partial x}\ , \qquad s \equiv \frac{\partial u}{\partial t}$$

$$\text{define } \ \vec{u}^{\,t} = (r, s) \qquad \vec{F}(\vec{u}) = \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} \cdot \vec{u}$$

$$\text{then } \ \frac{\partial \vec{u}}{\partial t} = -\frac{\partial \vec{F}(\vec{u})}{\partial x}$$

Simple Example

Take the simplest flux-conservative equation - u scalar

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

Forward time, centered space FTCS technique:

$$\frac{u(j,n+1)-u(j,n)}{\Delta t} = -c\left(\frac{u(j+1,n)-u(j-1,n)}{2\Delta x}\right)$$

This method turns out to be unstable. Why ?

von Neumann Stability

Imagine that the coefficients of the difference equation are so slowly varying that we can consider them constant in space and time. Then, the independent solutions of

$$\frac{u(j,n+1)-u(j,n)}{\Delta t}=-c\left[\frac{u(j+1,n)-u(j-1,n)}{2\Delta x}\right]$$

are of the form $u(j,n)=\xi^n e^{ikj\Delta x}$

Where k is real and ξ generally complex. Substituting into the equation above yields:

$$\xi(k)=1-i\frac{c\Delta t}{\Delta x}\sin k\Delta x$$

The magnitude of ξ>1 for all k≠0, so the method is unstable. If ξ<1, then the magnitude of the wave will decrease with time.

Lax Method

The problem with the FTCS algorithm is cured by Lax by making the following substitution:

$$u(j,n) \rightarrow \frac{1}{2}\left(u(j+1,n)+u(j-1,n)\right)$$

This yields

$$u(j,n+1)=\frac{1}{2}\left(u(j+1,n)+u(j-1,n)\right)-\frac{c\Delta t}{2\Delta x}\left(u(j+1,n)-u(j-1,n)\right)$$

Substituting $u(j,n)=\xi^n e^{ikj\Delta x}$

gives $\xi=\cos k\Delta x - i\dfrac{c\Delta t}{\Delta x}\sin k\Delta x$

The stability criterion implies $|\xi|^2 \leq 1 \quad \rightarrow \quad \dfrac{|c|\Delta t}{\Delta x}\leq 1$ **Courant condition**

as claimed. So, the Lax algorithm can be made stable.

Lax Method

Why does the Lax method work ? Rewrite

$$\frac{u(j,n+1)-u(j,n)}{\Delta t} = -c\left(\frac{u(j+1,n)-u(j-1,n)}{2\Delta x}\right)+\frac{1}{2}\left(\frac{u(j+1,n)+u(j-1,n)-2u(j,n)}{\Delta t}\right)$$

Which is the FTCS representation of

$$\frac{\partial u}{\partial t} = -c\,\frac{\partial u}{\partial x}+\frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 u}{\partial x^2}$$

A diffusion term is added to the equation which stabilizes the solution. Note that for $|\xi|^2 = \frac{|c|\Delta t}{\Delta x} < 1$ the amplitude of the wave will slowly decrease but this is usually not serious because Δx is very small (wavelengths < kΔx cannot be represented).

Wave Equation

The von Neumann analysis on the wave equation yields exactly the same Courant condition when using our finite difference technique:

$$y(i,n+1) = 2[1-r^2]y(i,n) - y(i,n-1) + r^2[y(i+1,n) + y(i-1,n)]$$

where

$$r \equiv \frac{c\Delta t}{\Delta x}$$

i.e., you have to use $r \leq 1$.

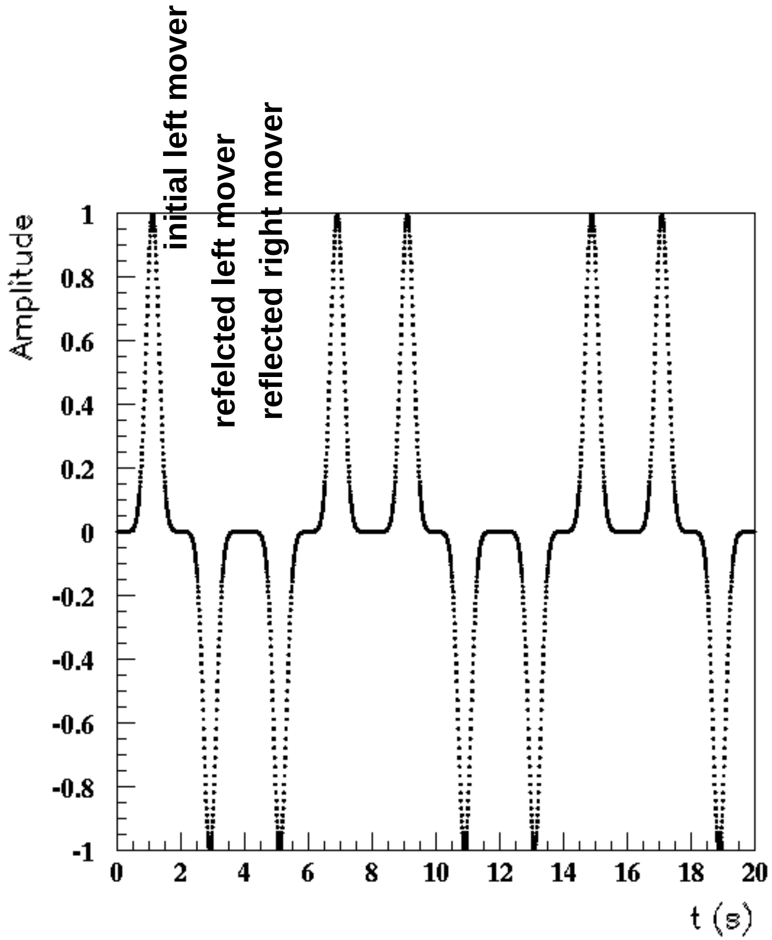
Back to our pulse on the string. Try a few different things:

1. The string is made of two parts with different velocities (e.g., different mass densities)
2. The ends are free - not tied down.

→ See 'movie'

Frequency Spectrum

Consider now a fixed point on the string and see how the amplitude evolves with time. We go back to the string with fixed ends for this:



This is for a point 1/4 of the distance from the left edge of the string. Let’s extract the power spectrum from a Fourier analysis (see last lecture).

Frequency Spectrum

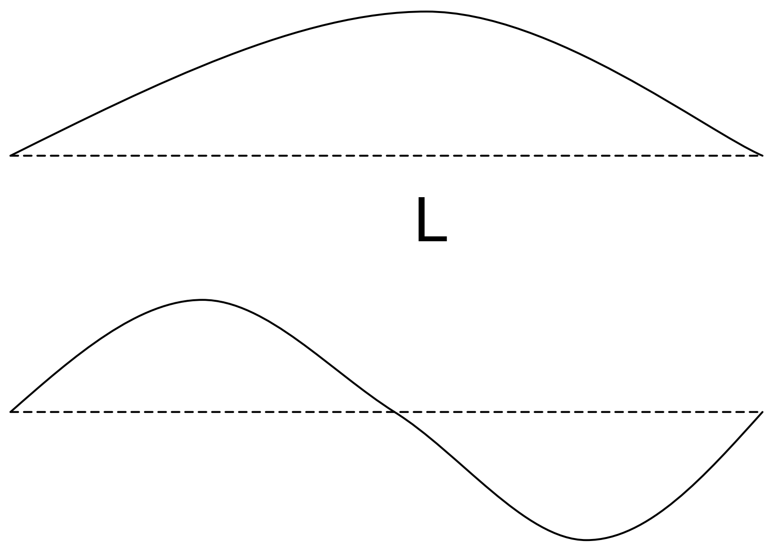
Recall:

$$P(0) = P(f_0) = \frac{1}{N^2}\left|C_0\right|^2$$

$$P(f_k) = \frac{1}{N^2}\left[\left|C_k\right|^2 + \left|C_{N-k}\right|^2\right] \quad k = 1,2,\ldots,\left(\frac{N}{2}-1\right)$$

$$P(f_c) = P(f_{N/2}) = \frac{1}{N^2}\left|C_{N/2}\right|^2$$

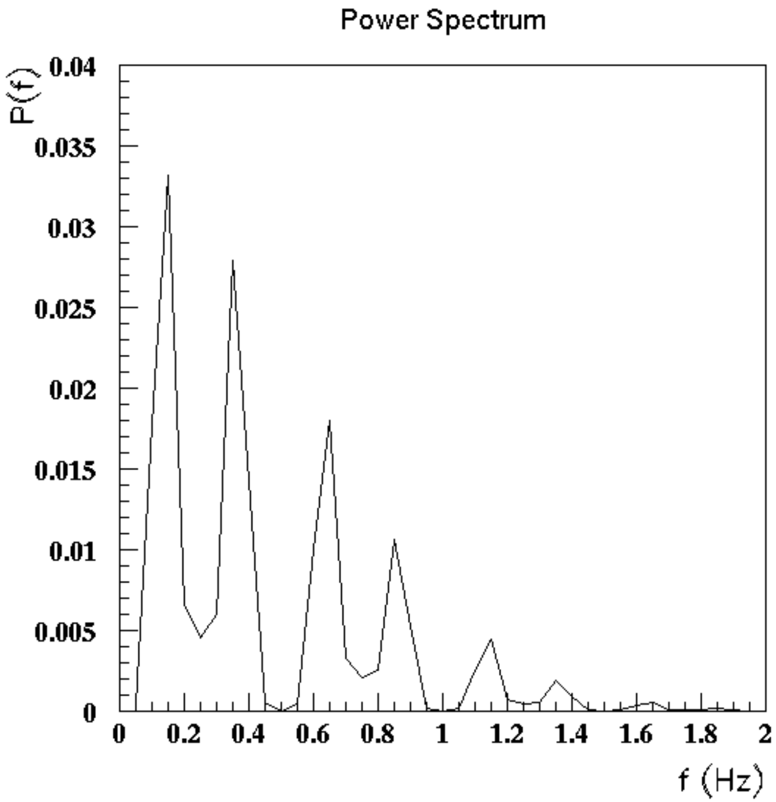
What do we expect ? String has length 4, so the expected frequencies are given by



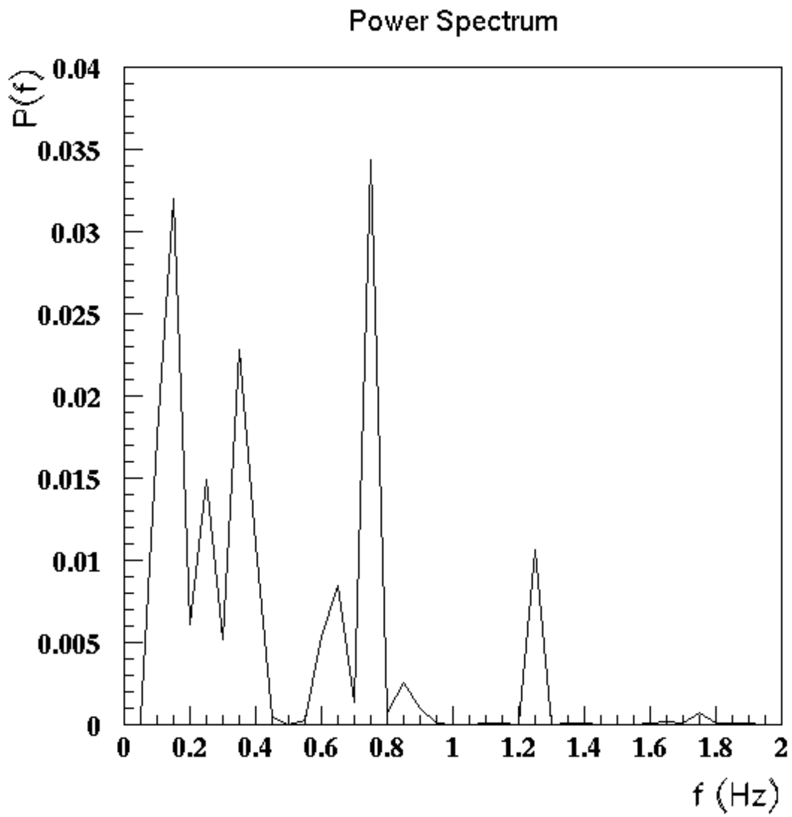
$$f = \frac{c}{\lambda} = N\frac{c}{(2L)} = N\frac{1}{8} \quad \text{(Hz)}$$

Frequency Spectrum

From Gaussian pulse at center



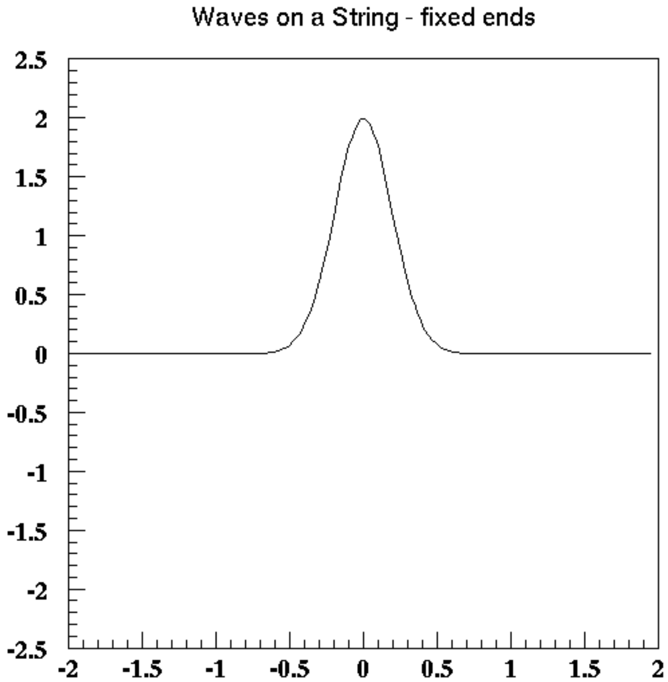
From displaced Gaussian pulse



See different multiples of fundamental frequency, but not all

Frequency Spectrum

Why are some of the frequency components missing ? Has to do with the symmetry of the starting wavepacket.



The wavepacket is symmetric about the center. The Fourier components will be preserved in time, so we can see which are present at the start. Clearly, only the frequency components corresponding to standing waves which are symmetric about 0 will survive.

Initial pulse makes a big difference in which frequency components present

$$f = \frac{c}{\lambda} = \mathrm{N}\frac{c}{(2L)}$$

$$\lambda = L, L/2, \ldots \text{ missing, or } \mathrm{N} = 2,4,\ldots$$

More Realistic String

For a more realistic string, we need to add some more effects, such as stiffness (force opposing the displacement in addition to tension) and damping due to frictional losses. Start with stiffness:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \rightarrow \quad \frac{\partial^2 y}{\partial t^2} = c^2 \left(\frac{\partial^2 y}{\partial x^2} - \varepsilon L^2 \frac{\partial^4 y}{\partial x^4} \right)$$

ε is the (dimensionless) stiffness parameter

L is the length of the string

The discrete approximation for the 4th derivative is

$$\frac{\partial^4 y}{\partial x^4} \approx \left(\frac{y(j+2,n) - 4y(j+1,n) + 6y(j,n) - 4y(j-1,n) + y(j-2,n)}{(\Delta x)^4} \right)$$

Some algebra yields

$$y(j,n+2) = [2 - 2r^2 - 6\varepsilon r^2 M^2] y(j,n) - y(j,n-1)$$

$$+ r^2 [1 + 4\varepsilon M^2][y(j+1,n) + y(j-1,n)] - \varepsilon r^2 M^2 [y(j+2,n) + y(i-2,n)]$$

where $M = L / \Delta x$

Stiff string

We have to decide how to handle the ends since now the grid points one removed from the end sees a fictitious grid point one step beyond the end (because of j±2) terms. Solution:

$$y(-1,n) = -y(+1,n) \quad y(M+1,n) = -y(M-1,n) \quad \text{(Hinge mechanism)}$$

Take some parameters from a typical grand piano

Note	F(Hz)	L(m)	C(m/s)	ε	b (s^{-1})
C2	65.4	1.9	250	$7.5\ 10^{-6}$	0.5
C4	262	0.62	330	$3.8\ 10^{-5}$	0.5
C7	2093	0.09	380	$8.7\ 10^{-4}$	0.5

Let's look at the power spectrum which results.

Stiff spring

From 'Numerical simulations of piano strings I. A physical model for a struck string using finite differences methods', A. Chaigne, A. Askenfelt, J. Acoust. Soc. Am. 95 (2) 1994,

find out there is a maximum number of spatial steps for numerical stability (after seeing my simulations diverge):

$$N_{\max} = \left\{ \frac{\left[-1 + \sqrt{1 + 16\varepsilon\gamma^2}\right]}{8\varepsilon} \right\}^{1/2} \quad \text{where} \quad \gamma = \frac{f_S}{2f_1}$$

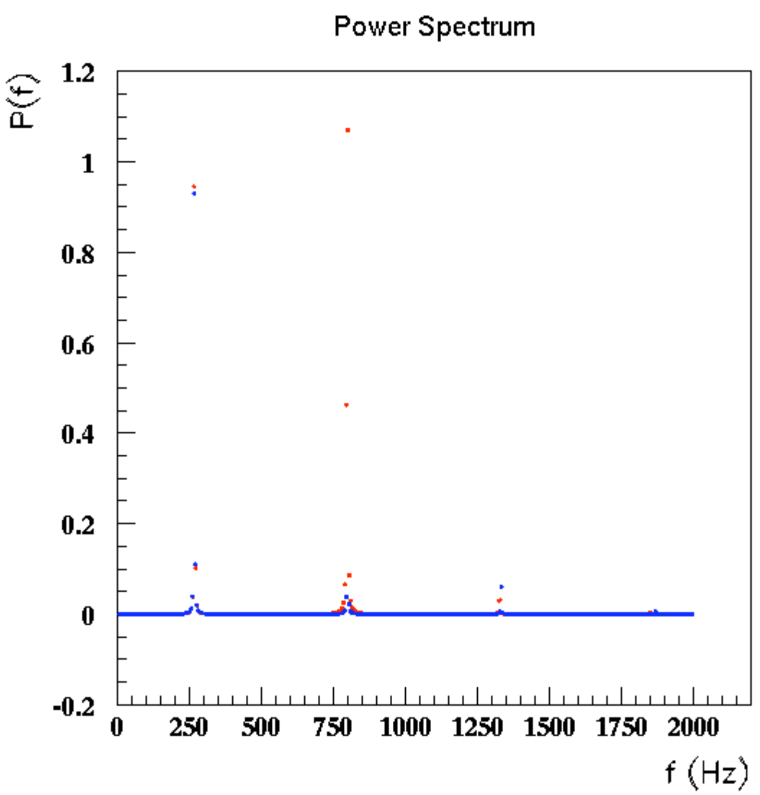
f_S is the sampling frequency

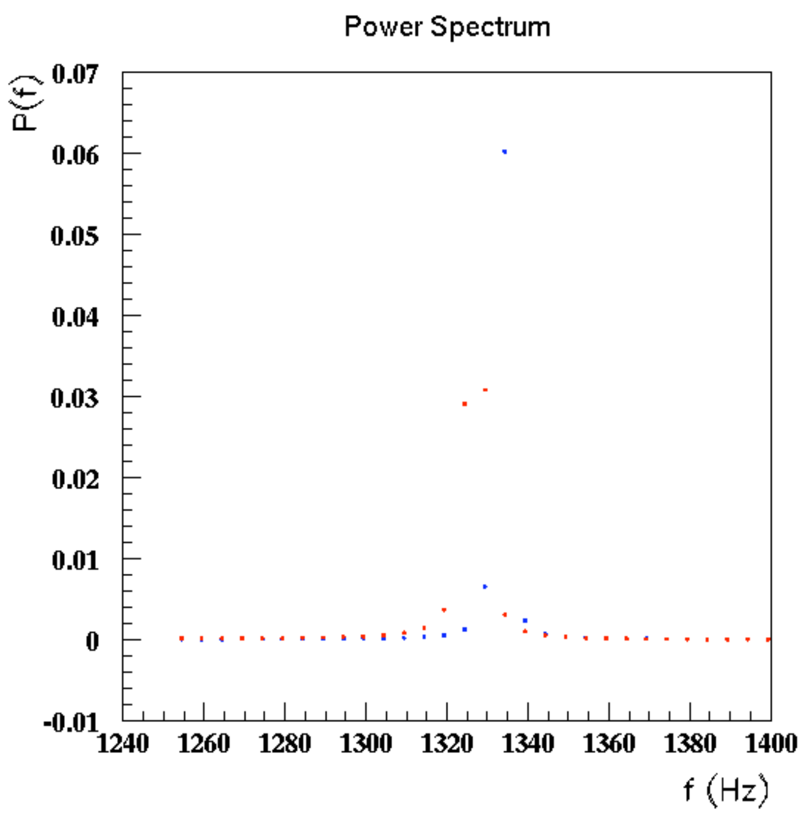
f_1 is the fundamental frequency

For our string, f_1=262 Hz, and f_s=10 kHz

Stiff spring

Comparison C2 from table (blue) with zero stiffness (red):





Dispersion is introduced in the stiff spring, because the effective wave speed becomes frequency dependent. The frequency components for the stiff string are shifted higher. Introduces characteristic sound.

Damping

Frictional forces will add some damping to the amplitude as a function of time. Chaigne, Askenfelt describe the full simulation of a piano string as follows:

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2} - \varepsilon L^2 \frac{\partial^4 y}{\partial x^4}\right) \underbrace{- 2b_1 \frac{\partial y}{\partial t} + 2b_3 \frac{\partial^3 y}{\partial t^3}}_{\text{Damping terms}} + \underbrace{f(x, x_0, t)}_{\text{Hammer excitation}}$$

Hammer excitation

Damping terms