Solving of linear Equations using SVD

- Solving a linear equation
- Gauss elimination and SVD
- HowTo
- Some tricks for SVD

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Problems in Linear Equations

- generally in solving $Ax=b$ two error contributions possible:
  1. intrinsic errors in $A$ and $b$
  2. numerical errors due to rounding

Mathematical description and handling available
General Error Analysis

Matrix Norm necessary
• e.g.

\[ \|A\|_2 = \max\{\lambda^{1/2}, \lambda \text{ Eigenwert von } A^T A \} \]

How does the final result (and its error) depend on the input value \( \tilde{A} = A + \delta A \) and \( \tilde{b} = b + \delta b \) ?

\[
\frac{\|\delta x\|}{\|x\|} \leq \frac{\text{cond}(A)}{1 - \text{cond}(A)\|\delta A\|/\|A\|} \left\{ \frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|} \right\}
\]

with

\[ \text{cond}_2(A) := \|A\|_2\|A^{-1}\|_2 = \frac{|\lambda_{\max}|}{|\lambda_{\min}|} \]

Eigenvalues of matrix
Example for Condition of Matrix

\[
\begin{bmatrix}
1.2969 & 0.8648 \\
0.2161 & 0.1441
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0.8642 \\
0.1440
\end{bmatrix}
\]

Exact solution: \((2,-2)^T\)

\[\text{cond}(A)=1.513\times10^8\]

Change input values:

\[0.8642 \rightarrow 0.86419999\]
\[0.1440 \rightarrow 0.14400001\]

approximate solution: \((0.9911,-0.4870)^T\)

**NOT ACCEPTABLE!**
Gauss Elimination

- Error due to rounding:

\[
\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq \frac{\text{cond}(A)}{1 - \text{cond}(A)\|\delta A\|_\infty/\|A\|_\infty} \{1.01 \cdot 2^{n-1}(n^3 + 2n^2) \text{eps}\}
\]

with \(\text{eps} = \text{machine accuracy (double: } \sim 10^{-16})\)

However, this is the worst case scenario!
Singular Value Decomposition

- here for (nxn) case, valid also for (nxm)
- Solution of linear equations numerically difficult for matrices with bad condition:
  - regular matrices in numeric approximation can be singular
  - \textbf{SVD} helps finding and dealing with the singular values
How does SVD work

Definition of singular value decomposition:

\[ U^T A V = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p) \]

with U and V orthogonal matrices

\[ Av_i = \sigma_i u_i, \]

\[ A^T u_i = \sigma_i v_i, \]

\[ A^T A v_i = \sigma_i^2 v_i \]

\( \sigma_i \) are eigenvalues of \( A^T A \)
Solution

Solution of the Equation:

\[ \bar{x} = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i \]

numeric behaviour of SVD can be determined:

\[ \|\delta A\|_2 \leq \text{eps} \|A\|_2 \]
SVD after Golub and Reisch

Householder Matrix $P,Q = 1 - w w^T$
(numerically OK)

$A = \begin{bmatrix}
  x & x & x & x \\
  x & x & x & x \\
  x & x & x & x \\
  x & x & x & x \\
\end{bmatrix}$

$PA = A' = \begin{bmatrix}
  * & * & * & * \\
  0 & * & * & * \\
  0 & * & * & * \\
  0 & * & * & * \\
\end{bmatrix}$

$A' = \begin{bmatrix}
  * & * & * & * \\
  0 & * & * & * \\
  0 & * & * & * \\
  0 & * & * & * \\
\end{bmatrix}$

$A'Q = A'' = \begin{bmatrix}
  x & * & 0 & 0 \\
  0 & * & * & * \\
  0 & * & * & * \\
  0 & * & * & * \\
\end{bmatrix}$

$P,Q$ unitary
SVD after Golub and Reisch

After a couple of iterations:

**J₀ Bidiagonal form:**

\[
J₀ = \begin{bmatrix}
q₁ & e₂ & 0 & 0 \\
0 & q₂ & e₃ & * \\
0 & 0 & q₃ & eₙ \\
0 & 0 & 0 & qₙ
\end{bmatrix}
\]

with:

\[
J₀ = Pₙ...P₁AQ₁...Q_{n−2}
\]

Q and P Householder matrices

A and J₀ have the same singular values
SVD after Golub and Reisch

Use iterative procedure to transform bidiagonal $J_0$ matrix to diagonal form.

Apply ‘Givens reflections’ to bidiagonal Matrix:

$$J_0 = S_{n-1,n} \cdots S_{23} S_{12} J_0 T_{12} T_{23} \cdots T_{n-1,n}$$

‘Givens Reflection’:

$\sim$ cubic convergence expected
Arithmetic Expenses

- Gauss (normal) solution
  - $\frac{1}{2} mn^2 + \frac{1}{6}n^3$

- SVD Golub-Reinsch
  - $2mn^2 + 4n^3$

- $m=n$:
  
  SVD is 9 times more expensive!
How to deal with Singularities

- singularities are determined with the SVD
- $1/\sigma_i$ is used for solution of linear equation
- relate $1/\sigma_i$ to machine accuracy and resolution $\tau$
  - usage of values $\sigma_i < \tau$ corrupts complete result!
    - careful handling necessary!
- simple approach:
  - neglect all values in matrix with $\sigma_i < \tau$
Smooth cutoff

- Regularisation of the singularities
- Replace the singular values with a function:

\[ \tau = 0.05 \]

\[ \frac{1}{s^2} \rightarrow \frac{s^2}{(s^2 + t^2)^2} \]
Example

- $d_i \sim b_i/\Delta b_i$
  - $d_i < 1 \rightarrow$ statistically insignificant

- Problem for alignment: determination of $\tau$
Conclusion

- Numeric Solution of a regular linear equation can be distorted by singular behaviour
- SVD returns singular values
- Singular values can be handled with smooth cut off
- Mathematical well described procedure